Excitation of phase patterns and spatial solitons via two-frequency forcing of a 1:1 resonance

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We show that a self-oscillatory system, driven at two frequencies close to that of the unforced system (resonance 1:1), becomes phase locked and exhibits two equivalent stable states of opposite phases. For spatially extended systems this phase bistability results in patterns characteristic for real order parameter systems, such as phase domains, labyrinths, and phase spatial solitons. In variational cases, the phase-locking mechanism is interpreted as a result of the periodic ''rocking'' of the system potential. Rocking could be tested experimentally in lasers and in oscillatory chemical reactions.

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The nature of patterns observed in dissipative nonlinear systems is deeply related with the degrees of freedom allowed to the system's phase. From this viewpoint one can distinguish between phase invariant and phase-locked systems, depending on whether the system's phase is free or it is attracted towards certain discrete values. The mathematical description of both types of systems is accordingly different. Phase invariant systems can be described by a complex field (the order parameter), and the complex Ginzburg-Landau and the complex Swift-Hohenberg equations are prototypical models for them [1]. Among phase-locked systems, especially relevant are those in which the phase is allowed to take only two values (differing by π) so that the system supports equivalent states of opposite signs. Hence, apart from a constant phase factor, they can be described by a real order parameter. Universal models for such systems are the real Ginzburg-Landau and the real Swift-Hohenberg equations [1]. A major difference between systems with complex order parameter (complex systems in the following) and systems with real order parameter (real systems) is the kind of defects supported by them. Vortices [2] are commonplace in complex systems, whereas they are not allowed in real ones. Similarly, real systems support domain walls [3] which on the contrary are unstable and decay into arrays of vortices in complex systems. Typical patterns of complex systems are ensembles of vortices and traveling waves, usually displaying a high degree of disorder. On the contrary, real systems tend to exhibit much more structured patterns, such as stripes (or labyrinths), hexagons, domain walls, and dark-ring solitons.

Here, we address the problem of how to ''convert'' an initially complex system into a real-like one via forcing in a 1:1 resonance. Apart from its interest from a fundamental viewpoint, this transformation could be useful for the excitation of phase domains and phase domain solitons in lasers and other phase invariant nonlinear optical systems, where such structures have potential applications for parallel information processing [4].

Although the mechanism we describe below is generalizable to other classes of complex systems, we focus on those displaying a spontaneous transition between a steady and an oscillatory state, both spatially uniform. This transition is technically termed a homogeneous Hopf bifurcation, and systems displaying it are complex since the phase of the oscillations is not fixed. A classical way to break this phase invariance is to submit the system to a *periodic* temporal forcing. This kind of forcing admits a universal description when the system is operated near the oscillation threshold, and forcing acts on a $n:m$ resonance, defined by the relation $\omega_e=(n/m)(\omega_0+\nu)$ between the external forcing frequency ω_e and the natural frequency of oscillations ω_0, where n/m is an irreducible integer fraction and ν is a small mistuning. In such case, the slowly varying complex amplitude of the oscillations A (the order parameter) verifies the following equation [5]:

$$\partial_t A=(\mu+i\nu)A+(1+i\alpha)\nabla^2 A-(1+i\beta)|A|^2 A + \mathcal{F}^m (A^*)^{n-1}, \tag{1}$$

where $\mathcal{F}$ is proportional to the amplitude of forcing, $\nabla^2 = \partial_x^2+\partial_y^2$, and A, and the space coordinates (x,y) have been normalized in order to make unity the diffusion and the saturating nonlinearity coefficients. In Eq. (1) all parameters are real and adimensional; μ is excitation parameter (proportional to the increase of the control parameter from its value at the bifurcation), α is the dispersion coefficient, and β is a nonlinear frequency shift coefficient. In the absence of forcing ($\mathcal{F}=0$) Eq. (1) reduces to the complex Ginzburg-Landau equation (CGLE), which is the universal description of self-oscillatory systems close to threshold [1,5,6]. Actual systems described by the CGLE are, e.g., self-oscillatory chemical reactions [7] and nonlinear optical cavities, such as lasers [8] and optical parametric oscillators [9]. A physical quantity, such as a concentration in a chemical reaction or the electric field in a laser, can be written in terms of the order parameter as $\mathrm{Re}(A(x,y,t)\exp[-i(\omega_0+\nu)t])$.

Next, we demonstrate that a self-oscillatory (complex) system can be transformed into a reallike one via resonant

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forcing. The idea is to drive the system with a spatially uniform, amplitude modulated forcing almost in resonance with the natural frequency of oscillations, which is a generalization of the classical periodic forcing in a 1:1 resonance to a quasiperiodic case. In particular, we consider a forcing whose amplitude is slowly modulated in time in the form $\mathcal{F}=\mathcal{F}_0\cos(\Omega t)$. Note that this is a two-frequency forcing since the actual forcing consists of two frequencies, $\omega_e \pm \Omega$ [10]. In this case, and close to the self-oscillation threshold, the system dynamics is governed by Eq. (1) with $m=n=1$ (1:1 resonance) and $\mathcal{F}=\mathcal{F}_0\cos(\Omega t)$ [11]. Upon defining $\sigma = \mathrm{sgn}(\mu)$, $\nu'=\nu/|\mu|$, $A'=A/\sqrt{|\mu|}$, $t'=|\mu|t$, $(x',y') = \sqrt{|\mu|}(x,y)$, $\omega=\Omega/|\mu|$, $F=\mathcal{F}_0/|\mu|^{3/2}$ (which is taken as real and positive without loss of generality), and removing the primes for the sake of not overburdening the notation, Eq. (1) becomes the following ac-driven CGLE:

$$\partial_t A=(\sigma+i\nu)A+(1+i\alpha)\nabla^2 A-(1+i\beta)|A|^2 A+F\cos(\omega t), \tag{2}$$

which is the object of our study [12].

In order to visualize the effect of the two-frequency forcing, we consider first the case of spatially uniform order parameter and vanishing α, β, and ν. Then the forced CGLE (2) is variational, $dA/dt=-\partial V/\partial A^*$, with potential $V=-\sigma|A|^2+\frac{1}{2}|A|^4-2F\cos(\omega t)\mathrm{Re}A$. In the absence of forcing ($F=0$), V is isotropic on the plane $\mathrm{Re}A-\mathrm{Im}A$; for $\sigma=+1$ ($\mu>0$), V has the shape of a sombrero, displaying a local maximum at the origin (the unstable trivial state $A=0$) and a degenerate minimum along the circle $|A|^2=1$ (the spatially uniform steady state with arbitrary phase). For a constant driving term ($F\neq 0$, $\omega=0$; the classical periodic driving in a 1:1 resonance) V becomes ''tilted'' and exhibits an isolated minimum on the axis $\mathrm{Re}A$. Finally, for a periodic forcing term ($F\neq 0$, $\omega\neq 0$; the considered two-frequency forcing) V undergoes a periodic tilting, or ''rocking,'' around the axis $\mathrm{Im}A$ and the system state vector $\mathbf{A}=(\mathrm{Re}A,\mathrm{Im}A)$ oscillates back and forth across that axis. Since for $\sigma=+1$ the origin is repulsive, $\mathbf{A}$ avoids in principle, that point and two equivalent trajectories (differing in the sign of $\mathrm{Im}A$) are allowed. Thus, two equivalent oscillating states of opposite phases are expected and, in principle, the initially complex system is transformed into a reallike system. Given the ''mechanical'' effect that the two-frequency forcing has on the system's potential, and in order to clearly identify it against the usual (single-frequency) periodic forcing (corresponding to $\omega=0$), we propose to term the forcing mechanism as ''rocking.''

Next, we show rigorously that the previous pictorial image holds even in the more general, nonvariational case. We consider the limit of ''strong and fast'' rocking ($F=f\omega, \omega \gg 1$) and keep f and the rest of parameters as $O(\omega^0)$ quantities [13]. This allows (i) to separate the slow time scale t of the unforced system from the fast time scale $\tau=\omega t$ of rocking ($\partial_t \to \omega\partial_\tau+\partial_t$), and (ii) to seek solutions to the forced CGLE (2) in series of ω of the form $A(x,y,t) = A_0(x,y,t,\tau)+O(\omega^{-1})$. At order ω, we find

$$A_0(x,y,t,\tau)=f\sin(\tau)+ia(x,y,t), \tag{3}$$

where $a(x,y,t)$ does not depend on the fast time scale. The evolution equation for $a(x,y,t)$ is found as a solvability condition at order ω^0 of the asymptotic expansion, which reads

$$\partial_t a=(\lambda+i\theta)a+(1+i\alpha)\nabla^2 a-(1+i\beta)|a|^2 a + \gamma(1+i\beta)a^*, \tag{4}$$

where $\lambda=\sigma-2\gamma$, $\theta=\nu-2\gamma\beta$, and $\gamma=\frac{1}{2}f^2=\frac{1}{2}(F/\omega)^2$ is the rocking parameter. To the leading order the actual order parameter reads $A(x,y,t)=\sqrt{2\gamma}\sin(\omega t)+ia(x,y,t)$ [Eq. (3)]. Note that the one-rocking-period average of the order parameter $\langle A(x,y,t)\rangle \equiv 1/T\int_t^{t+T}dt'A(x,y,t')\simeq ia(x,y,t)$, $T=2\pi/\omega$, in this fast rocking limit.

Equation (4) is a CGLE with phase sensitive gain (last term), which favors a phase locking as it imposes the discrete phase symmetry $a\to -a$. Thus, two equivalent states of oscillation of opposite phases exist. We note that Eq. (4) is similar to that describing oscillatory systems periodically forced at *twice* their natural frequency of oscillation (2:1 resonance), i.e., Eq. (1) with $m=1$ and $n=2$ [5].

For the sake of simplicity, we limit in the following to the case $\beta=0$:

$$\partial_t a=(\sigma-2\gamma+i\nu)a+(1+i\alpha)\nabla^2 a-|a|^2 a+\gamma a^*. \tag{5}$$

Equation (5) has two relevant spatially homogeneous solutions (apart from the trivial one $a=0$) given by $a=\pm|a|\exp i\varphi$, $|a|^2=\sigma-2\gamma+\sqrt{\gamma^2-\nu^2}$, and $\sin 2\varphi=\nu/\gamma$. Note that below the bifurcation ($\mu<0$, i.e., $\sigma=-1$) $|a|^2<0$, so that rocking requires $\sigma=+1(\mu>0)$ to be effective. Hence, we consider in the following $\sigma=+1$. These ''rocked states'' have equal intensities $|a|^2$ but opposite signs. They exist for $|\nu|<\gamma<\frac{1}{3}(2+\sqrt{1-3\nu^2})$ that requires $\nu^2<1/3$. The linear stability analysis of both the trivial and the rocked states of Eq. (5) shows that they are unaffected by pattern forming instabilities for $\alpha\nu\leqslant 0$. For $\alpha\nu>0$ they can become unstable against spatial modulations of wave number k given by $\alpha k_{triv}^2=\nu-\mathrm{sgn}(\alpha)\gamma/\sqrt{1+\alpha^2}$ and $\alpha k_{rock}^2=\nu-\mathrm{sgn}(\alpha)\sqrt{(1-2\gamma)^2+\nu^2}/\sqrt{1+\alpha^2}$, respectively (details will be given elsewhere). Thus, for $\alpha\nu\leqslant 0$ there exists bistability between the two oppositely phased spatially homogeneous rocked states, which enables the existence of domain walls connecting them. Outstandingly in the variational case ($\alpha=\nu=0$) two kinds of such solutions to Eq. (5) for $\sigma=+1$ are analytical in one spatial dimension, and are known as Ising (I) and Bloch (B) walls [14]

$$a_I(x)=\pm g\tanh(gx/\sqrt{2}), \tag{6}$$

$$a_B(x)=\pm[g\tanh(\sqrt{2\gamma}x)\pm i\sqrt{1-5\gamma}\,\mathrm{sech}(\sqrt{2\gamma}x)], \tag{7}$$

where $g=\sqrt{1-\gamma}$. The Ising wall (6) shows up as a dark line (where $a_I=0$) and is stable for $1/5<\gamma<1$. The Bloch wall (7), a gray line, is stable for $0<\gamma<1/5$. At $\gamma=1/5$ a bifurcation takes places and an Ising-Bloch transition is observed. Both structures persist in nonvariational cases, at least for small α and ν, and the transition between them still occurs at

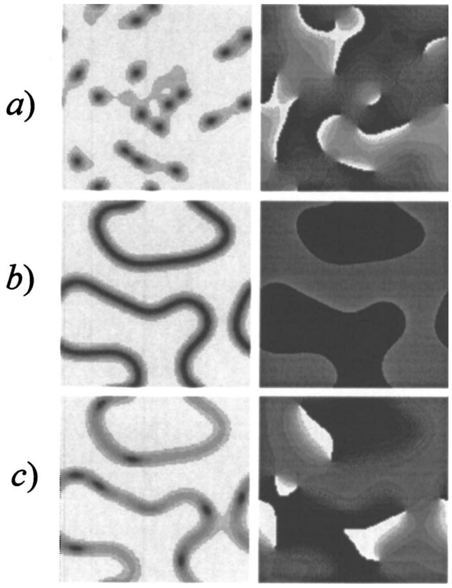

FIG. 1. Phase domains in the variational case ($\alpha=\beta=\nu=0$) as obtained by numerical integration of the ac-driven CGLE (2) for $\sigma=+1$ and $\Omega=4\pi$. The side of the square integration region had a length $L=58$. Left column, intensity of the one-rocking–period average of A (black and white correspond to zero and maximum intensity, respectively). Right column, phase of the averaged order parameter. (a) $F=0$ ($\gamma=0$), vortex ensemble at $t=10$. (b) $F=10$ ($\gamma=0.317$), Ising walls at $t=30$. (c) $F=5$ ($\gamma=0.079$), Bloch walls at $t=30$.

$\gamma=1/5$ [14]. Differently from the variational case, Bloch walls now move with a velocity proportional to their chirality [14].

These analytical predictions were tested against the numerical integration of the forced CGLE (2), with $\sigma=+1$ [15]. In order to give evidence of the robustness of rocking, we show next results for a rocking frequency $\omega=4\pi$, a value which is not extremely larger than unity.

First, we consider the variational case ($\alpha=\beta=\nu=0$). Without forcing ($F=0$) vortices are spontaneously formed [Fig. 1(a)]. For large driving amplitude, vortices are substituted by domains, separated by dark lines (Ising walls) [Fig. 1(b)]. For weaker forcings, lines become gray (Bloch walls) [Fig. 1(c)] indicating that the order parameter does not vanish at the interface [Eq. (7)]. In the Bloch regime phase locking is not perfect and walls tend to contain vortices (black dots in the figure), especially for weak driving. Our numerics also confirm the relevance of the rocking parameter $\gamma=\frac{1}{2}(F/\omega)^2$. For instance, for $\omega=4\pi$ (the value used in Fig. 1) the Ising-Bloch transition occurred for $F=8.0\pm0.1$($\gamma=0.203\pm0.005$), which is in a very good agreement with the analytical prediction $\gamma=1/5$. The upper boundary of existence of Ising domains was $F=16.5\pm0.1$ ($\gamma=0.862\pm0.010$), which agrees [within an $O(\omega^{-1})$ error] with the analytical prediction $\gamma=1$. We also checked that these threshold values for F scale linearly with ω, in agreement with the functional dependence of γ on them. We remark that phase domains *in two spatial dimensions* are transient solutions of the forced CGLE (2) in this variational case, due to curvature effects domains contract until a single phase ultimately dominates.

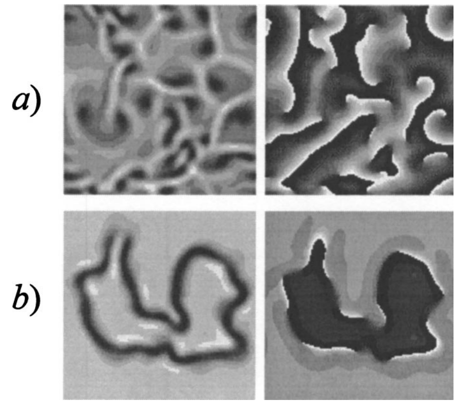

FIG. 2. Phase domains in the nonvariational case $\alpha=10$. $L=183$. The rest of the parameters are as in Fig. 1. (a) $F=0$ ($\gamma=0$), vortex ensemble with shocks at $t=60$. (b) $F=7$ ($\gamma=0.155$), Bloch walls at $t=70$.

Finally, we show results corresponding to the essentially nonvariational case $\alpha=10$ ($\alpha\gg1$ is typical of nonlinear optics). The null detuning case $\nu=0$ is shown in Fig. 2. Again, without driving a chaotic ensemble of vortices is observed [Fig. 2(a)]. Typical for this predominantly dispersive case is that vortices are separated by shock waves. For sufficiently large rocking strength phase domains separated by Ising-like walls appear similarly to the variational case (not shown). For a weaker rocking Bloch walls are observed [Fig. 2(b)]. Differently from the variational case Bloch walls need not be closed, but can end abruptly. These walls are highly unstationary exhibiting random evolution (snaking instabilities, breakings, and reconnections) which is probably due to the intrinsic motion of the Bloch wall related to its chirality in nonvariational cases. Here, like in the variational case, both Ising and Bloch domains contract and eventually disappear. For $\alpha\nu>0$, our simulations also confirm that extended patterns can be excited. Ising-like labyrinths [Fig. 3(a)] as well as Bloch-like labyrinths [Fig. 3(b)] are formed depending on

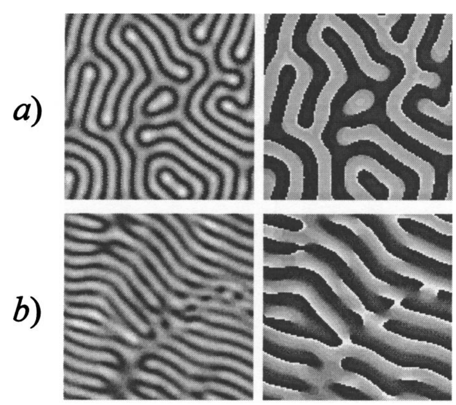

FIG. 3. Labyrinths obtained for $\nu=0.6$. The rest of the are parameters as in Fig. 2. Plots correspond to $t=170$. (a) $F=13$ ($\gamma=0.54$), Ising-like labyrinth. (b) $F=7.5$ ($\gamma=0.178$), Bloch-like labyrinth.

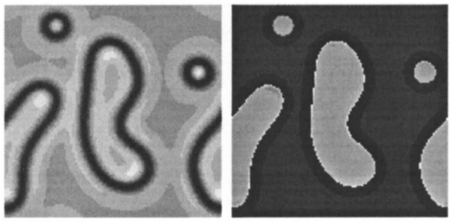

FIG. 4. Spatial solitons obtained for $\nu = 0.27$ and $F = 10$ ($\gamma = 0.317$). The rest of the parameters are as in Fig. 3. Two stable solitons and two contracting domains are shown at $t = 360$. At $t = 1200$ contraction ceases and the final distribution for this particular initial condition is four stable ring solitons (not shown).

the rocking strength. In the Bloch case the gray lines are reconnected in some places and also some vortices are visible, indicating that phase is allowed to display smooth variations, although it is still partially quenched. Finally, a most outstanding result is obtained for values of ν between those for labyrinths and those for contracting domains; the formation of spatial solitons (Fig. 4).

The transformation of a complex system into a reallike one via rocking, as evidenced in this paper, can be especially useful in nonlinear optics where phase domains and bistable phase solitons could be applied for the purposes of parallel information processing. This is especially relevant in the case of lasers, which can be driven at frequencies close to that of lasing (the usual laser with injected signal), i.e., in their 1:1 resonance, but they are insensitive to drivings on other resonances. All these ideas could be well tested also in other self-oscillatory systems suitable to be driven on resonance, like the Belousov-Zhabotinsky reaction [16].

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[11] Here, we have introduced the mathematical description of two-frequency forcing of a self-oscillatory system within its 1:1 resonance as a generalization of the corresponding perfectly periodic forcing. Alternatively one can show, making use of standard multiple scale techniques, that Eq. (1) with $m = n = 1$ and $\mathcal{F} = \mathcal{F}_0 \cos(\Omega t)$, is the generic description of a self-oscillatory system submitted to a two-frequency forcing. This demonstration requires that $\mu = O(\varepsilon^2)$, $\nu = O(\varepsilon^2)$, $\Omega = O(\varepsilon^2)$, and $\mathcal{F}_0 = O(\varepsilon^3)$, being $0 < \varepsilon \ll 1$ a smallness parameter [12].

[12] Note that in Eq. (2) all quantities are of order ε^0 due to normalization.

[13] Note that this ‘‘strong and fast’’ forcing limit is not at odds with the scalings used to derive the forced CGLE. In fact, the inequality $\omega \gg 1$ (i.e., $\Omega \gg |\mu|$) does not violate the assumptions $\mu = O(\varepsilon^2)$, $\Omega = O(\varepsilon^2)$, [11], whenever $\omega \ll \varepsilon^{-1}$. An analogous reasoning holds for the scaling of F. In any case, as we show below by direct numerical simulation, the assumption of strong and fast forcing is just a formal necessity in order to capture analytically the effects of forcing, but not a true requirement for observing the phenomena we later describe.

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